FRACTAL WEYL LAW FOR SKEW EXTENSIONS OF EXPANDING MAPS

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ABSTRACT. We consider compact Lie groups extensions of expanding maps of the circle, essentially restricting to U(1) and SU(2) extensions. The central object of the paper is the associated Ruelle transfer (or pull-back) operator $\hat{F}$. Harmonic analysis yields a natural decomposition $\hat{F} = \oplus \hat{F}_\alpha$, where α indexes the irreducible representation spaces. Using Semiclassical techniques we extend a previous result by Faure proving an asymptotic spectral gap for the family $\left\{ \hat{F}_\alpha \right\}$ when restricted to adapted spaces of distributions. Our main result is a fractal Weyl upper bound for the number of eigenvalues (the Ruelle resonances) of these operators out of some fixed disc centered on 0 in the complex plane.

CONTENTS

1. Introduction

Partially hyperbolic dynamical systems form a class of chaotic models of a more subtle type than the ones provided by the well understood uniformly hyperbolic setting [1, 2]. The difficulty in treating such systems is caused by the presence of a neutral bundle that can drastically slow (or even prevent) the escape towards statistical equilibrium. Compact Lie groups extension are an attractive class of models of such type (the fibers of the neutral bundle are homeomorphic to a given compact Lie group and the system acts isometrically between fibers), where representation theory and Lie groups techniques can help tackle issues such as mixing rates or stable ergodicity.

In this article we focus on skew extensions of expanding maps of the circle. Put $S^1 \equiv \mathbb{R}/\mathbb{Z}$ and let $E : S^1 \to S^1$ be a $C^\infty-$expanding map with $k > 1$ smooth inverse branches E_ϵ^{-1} $\epsilon = 0, ..., k-1$. This map is automatically topologically mixing [15] and mixing w.r. to a smooth invariant absolutely continious measure μ called the SRB measure. Let $\mathbb{G}$ be a compact Lie group with normalized Harr measure m. For any smooth map $\tau : S^1 \to \mathbb{G}$ the skew extension $\widehat{E_\tau} : S^1 \times \mathbb{G} \to S^1 \times \mathbb{G}$ is defined by

$$(1.1) \qquad \widehat{E_\tau}(x, g) = (E(x), \tau(x)g).$$

The measure $\hat{\mu}_\tau := \mu \times m$ is an invariant smooth absolutely continuous measure for $\widehat{E_\tau}$. For any two reasonably regular observables Ψ, Φ on $S^1 \times \mathbb{G}$ the central object in ergodic theory is the correlation function, defined for any $n \in \mathbb{N}^*$ by

$$(1.2) \qquad C_{\Psi,\Phi}(n) := \left(\Psi \circ \widehat{E_\tau}^n ; \Phi \right)_{L^2(\hat{\mu}_\tau)}$$

which converges to $\int \bar{\Psi} d\hat{\mu}_\tau \int \Phi d\hat{\mu}_\tau$ as $n \to \infty$ iff $\widehat{E_\tau}$ is mixing. Fundamental questions concern the rate of decay of $C_{\Psi,\Phi}(n) - \int \bar{\Psi} d\hat{\mu}_\tau \int \Phi d\hat{\mu}_\tau$ and are related to the spectral properties of the so-called *Ruelle transfer operator*

$$(1.3) \qquad \widehat{F_\tau} : \Psi \mapsto \Psi \circ \widehat{E_\tau}, \ \Psi \in C^\infty \left(S^1 \times \mathbb{G} \right)$$

in adapted Banach spaces of distribution [21, 23]. The first quantitative results in this context where obtained by Dolgopyat [4] who showed an exponential decay rate (exponential mixing) for generic maps τ. On the other hand Naud [18, 19], in the analytic context showed that the rate of mixing cannot exceed a certain bound related to the topological pressure of $-2 \log |E'|$. In the Abelian case $\mathbb{G} \equiv U(1)$ Faure [7], using semi-classical analysis showed that the transfer operator acting in some Hilbert spaces of distributions generically exhibits an essential spectral radius bounded by $1/\sqrt{E_{\min}}$; with $E_{\min}$ the minimal expansion rate of E. This results already deduced by Tsujii [29] in the setting of suspension semi-flows, shows that up to terms of order ρ^n, $1 > \rho > 1/\sqrt{E_{\min}}$ the escape towards equilibrium is governed by a linear finite rank operator (theorem 5 in [7] and Eq.(1) in [29]). We

will show that this result extends to the simplest non Abelian compact lie group $\mathbb{G} \equiv \mathrm{SU}(2)$ (Theorem 2 and Corollary 3).

As mentioned above to treat such models one uses harmonic analysis. In particular the celebrated Peter-Weyl theorem [26] gives

$$(1.4) \qquad L^2(\mathbb{G}) = \oplus_\alpha \dim_\mathbb{C}(\mathcal{D}_\alpha)\, \mathcal{D}_\alpha,$$

where $\mathcal{D}_\alpha$ are finite dimensional irreducible hermitian vector spaces of representation for $\mathbb{G}$. For $\mathbb{G} \equiv \mathrm{U}(1)$ this is simply the Fourier decomposition of functions. The transfer operator (1.3) extends to a continuous operator on $L^2(S^1 \times \mathbb{G}) = L^2(S^1) \otimes L^2(\mathbb{G})$ and preserves the decomposition induced by (1.4), so that

$$(1.5) \qquad \hat{F}_\tau = \oplus_\alpha \dim_\mathbb{C}(\mathcal{D}_\alpha)\, \hat{F}_\alpha$$

with $\hat{F}_\alpha := \hat{F}_\tau|_{L^2(S^1) \otimes \mathcal{D}_\alpha}$ acting on smooth vector valued functions $\boldsymbol{\varphi} : S^1 \to \mathcal{D}_\alpha$ as

$$(1.6) \qquad \left(\hat{F}_\alpha \boldsymbol{\varphi}\right)(x) = \hat{\tau}_\alpha(x)\boldsymbol{\varphi}(x),$$

with $\hat{\tau}_\alpha(x)$ the representation in $\mathcal{D}_\alpha$ of $\tau(x) \in \mathbb{G}$. In some standard Sobolev spaces of distributions these operators have discrete spectrum (the Ruelle spectrum of resonances, Theorem 1). For the trivial representation corresponding to $\dim_\mathbb{C}(\mathcal{D}_\alpha) = 1$ and $\hat{\tau}_\alpha(x) \equiv \mathrm{Id}$, the constant function is an obvious eigenfunction of $\hat{F}_\alpha$ with eigenvalue one[1].

For simplicity we shall restrict ourselves to the cases $\mathbb{G} \equiv \mathrm{U}(1)$ and $\mathbb{G} \equiv \mathrm{SU}(2)$, respectively the Abelian and the simplest non-Abelian compact Lie groups. For $\mathbb{G} \equiv \mathrm{U}(1)$, $\alpha = \nu \in \mathbb{Z}$ and $\hat{\tau}_\nu(x) = e^{i\nu\Omega(x)}$; $\Omega \in C^\infty(S^1)$. For $\mathbb{G} \equiv \mathrm{SU}(2)$, $\alpha = j \in \frac{1}{2}\mathbb{N}$ and $\dim_\mathbb{C}(\mathcal{D}_j) = 2j + 1$. The point we wish to make here is that the spectral study of the familly $\left\{\hat{F}_\alpha\right\}$ is a well-posed semi-classical problem, as in [7] when $\mathbb{G} \equiv \mathrm{U}(1)$. To see this in the non Abelian setting we will use Lie groups coherent states theory [20]. Doing so we will derive a Fractal weyl asymptotic for the number of resonances outside a fixed spectral radius (Theorem 5) in the (semi-classical) limit ν or $j \to \infty$. As in the previous papers by Faure [9, 8, 7], the techniques used are derived from the works of Sjöstrand [24] Zworski-Lin-Guillope [13], Sjöstrand-Zworski [25], in the context of chaotic scattering.

2. Statement of the results

The operators $\hat{F}_\alpha$ defined in (1.6) extend to the distribution spaces $\mathcal{D}'(S^1) \otimes \mathcal{D}_\alpha$ by setting, for any $\boldsymbol{\psi} \in \mathcal{D}'(S^1) \otimes \mathcal{D}_\alpha$, $\boldsymbol{\varphi} \in C^\infty(S^1) \otimes \mathcal{D}_\alpha$: $\left(\hat{F}_\alpha \boldsymbol{\psi}\right)(\bar{\boldsymbol{\varphi}}) := \boldsymbol{\psi}\left(\overline{\hat{F}_\alpha^* \boldsymbol{\varphi}}\right),$

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[1]For other α they might not be any eigenvalues on the unit circle. However if τ maps S^1 into a closed subgroup H of $\mathbb{G}$ or whenever τ is co-homologous to such a map, meaning that $\tau = \eta - \eta \circ E$ with $\eta : S^1 \to H \subset \mathbb{G}$, Then they will be some eigenvalues on the unit circle as $\widehat{E_\tau}$ is not topologically transitive in this case, hence not weakly-mixing [15].
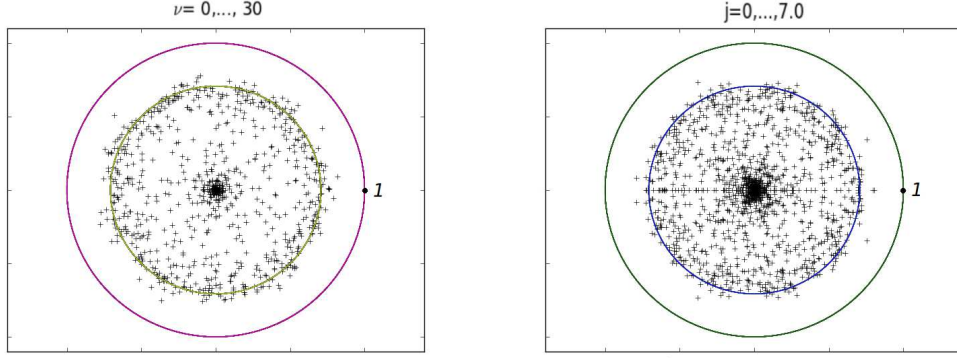
FIGURE 2.1. Numerical computation of the superposition of the resonance spectrum of $\hat{F}_\alpha$ (1.6) for to the first few values of α, for particular skew extension of the linear map $E(x) = 2x \bmod 1$. On the left $\tau(x) = \frac{1}{2\pi} \cos(2\pi x)$ seen as an element of the Abelian group $\mathbb{R}/\mathbb{Z}$. On the right $\tau(x) = e^{i \cos(2\pi x) J_3} e^{i\theta J_2} e^{i \cos(2\pi x) J_3} \in \mathrm{SU}(2)$ with $iJ_l; l = 1, 2, 3$ the generators of $\mathfrak{su}(2)$ and $\theta \neq 0$ a fixed arbitrary value. In both pictures the inner circle corresponds to the asymptotic gap of Theorem 2 and the black dot is the dominant simple eigenvalue $\lambda = 1$.

with the L^2-adjoint $\hat{F}_\alpha^*$ given by

$$(2.1) \qquad \left(\hat{F}_\alpha^* \boldsymbol{\varphi} \right)(x) = \sum_{y \in E^{-1}\{x\}} \frac{1}{E'(y)} \hat{\tau}_\alpha(y) \boldsymbol{\varphi}(y).$$

Recall that for $m \in \mathbb{R}$ the Sobolev spaces $H^m(S^1) \subset \mathcal{D}'(S^1)$ consists of distributions ψ (or continious functions if $m > 1/2$) whose fourier series $\hat{\psi}(\xi)$ satisfy $\|\psi\|_{H^m} := \sum_{\xi \in 2\pi\mathbb{Z}} \left| \langle \xi \rangle^m \hat{\psi}(\xi) \right|^2 < \infty$, with $\langle \xi \rangle := (1 + \xi^2)^{1/2}$. It can equivalently be written [26]:

$$(2.2) \qquad H^m(S^1) := \left\langle \hat{\xi} \right\rangle^{-m} \left(L^2(S^1) \right), \ \hat{\xi} := -i\frac{d}{dx}.$$

$\left\langle \hat{\xi} \right\rangle^m$ is a typical representative of the class of Pseudo-Differential-Operators (PDO) of order m (cf section 3 with $\hbar = 1$).

Theorem 1. *(Ruelle [22], Faure [7]). Here $\mathbb{G}$ can be any compact Lie group and $\hat{F}_\alpha$ is defined by (1.5) and (1.6). Then $\forall m, \forall \alpha$, the operator $\hat{F}_\alpha$ acts in $H^m(S^1) \otimes \mathcal{D}_\alpha$ and has discrete spectrum outside a disc of radius $r_m := e_{\min}^m (k/e_{\min})^{1/2}$, with $e_{\min} := \min_x E'(x) > 1$. The generalized eigenvalues outside this disc, along with they respective eigenspaces, do not depend on m and define the Ruelle spectrum of resonances of $\hat{F}_\alpha$. See figure 2.1.*

Proof. Let $\mathbb{G}$ be any compact Lie group and $\mathcal{D}_\alpha$ be some irreducible representation space for $\mathbb{G}$. Fix $m \in \mathbb{R}$ (we will write H^m, resp. L^2, for $H^m(S^1)$ and $L^2(S^1)$) and define $\mathcal{F}_E : \varphi \mapsto \varphi \circ E$ and $(\hat{\tau}\boldsymbol{\varphi})(x) := \hat{\tau}_\alpha(x)\boldsymbol{\varphi}(x)$, so that $\hat{F}_\alpha = \hat{\tau}(\mathcal{F}_E \otimes \mathbb{I})$. Recall from [7] that $\mathcal{F}_E$ restricts to $\mathcal{F}_E : H^m \to H^m$ and has its essential spectral radius bounded by r_m. Thus $\tilde{Q}_m := \left\langle \hat{\xi} \right\rangle^m \mathcal{F}_E \left\langle \hat{\xi} \right\rangle^{-m}$ is L^2-continuous with the same essential spectral radius estimate. Define $\hat{A}_m := \left\langle \hat{\xi} \right\rangle^m \otimes \mathbb{I}$ so that $\hat{A}_m^{-1}(L^2 \otimes \mathcal{D}_\alpha) = H^m \otimes \mathcal{D}_\alpha$. Consider $\hat{Q}_m = A_m \hat{F}_\alpha A_m^{-1}$ and $\hat{P} = \hat{Q}_m^* \hat{Q}_m$. We see that $\hat{P} = \hat{A}_m^{-1} \hat{B}_{2m} \hat{A}_m^{-1}$ with $\hat{B}_{2m} := \hat{F}_\alpha^* \hat{A}_m^2 \hat{F}_\alpha = (\mathcal{F}_E^* \otimes \mathbb{I}) \hat{\tau}^{-1} \hat{A}_m^2 \hat{\tau} (\mathcal{F}_E \otimes \mathbb{I})$. Commuting $\hat{A}_m^2$ and $\hat{\tau}$ we get that $\hat{B}_{2m} = \mathcal{F}_E^* \left\langle \hat{\xi} \right\rangle^{2m} \mathcal{F}_E \otimes \mathbb{I} + \hat{C}_{2m-1}$, with $\hat{C}_{2m-1}$ a matrix whose entries are PDOs of order $2m-1$ (cf section 3 with $\hbar = 1$). Therefore $\hat{P} = \tilde{Q}_m^* \tilde{Q}_m \otimes \mathbb{I} + \hat{C}_{-1}$ with $\hat{C}_{-1}$ a matrix whose entries are PDOs of order -1 hence compact. The independence in the value of m for the discrete eigenvalues is proven in [9], and is a consequence of the fact that $H^{m'}$ is dense in H^m for any $m' \geq m$. $\square$

Theorem 2. *(Tsujii [29] and Faure [7] for the Abelian case). Let $\mathbb{G}$ be either $U(1)$ or $SU(2)$. Recall that in these cases α denotes respectively $\nu \in \mathbb{Z}$ or $j \in \frac{1}{2}\mathbb{N}$. For $m < 0$ sufficiently negative, if the map $\widehat{E_\tau}$ (1.1) is partially captive (definition 16) then the spectral radius $\hat{F}_\alpha : H^m(S^1) \otimes \mathcal{D}_\alpha \to H^m(S^1) \otimes \mathcal{D}_\alpha$ satisfies in the semiclassical limit $\alpha \to \infty$*

$$(2.3) \qquad r_s\left(\hat{F}_\alpha\right) \leq \frac{1}{\sqrt{E_{\min}}} + o(1),$$

with $E_{\min} := \lim_{n\to\infty} \left(\min_x (E^n)'(x)\right)^{1/n} > e_{\min}$ the minimal expansion rate of E. Also for any $\rho > \frac{1}{\sqrt{E_{\min}}}$ there exists n_0, $\alpha_0 > 0$, $m_0 < 0$ s.t. $\forall |\alpha| \geq \alpha_0$, $m \leq m_0$ $\left\| \hat{F}_\alpha^{n_0} \right\|_{H^m_{1/\alpha} \otimes \mathcal{D}_\alpha} \leq \rho^{n_0}$, where $\|\cdot\|_{H^m_{1/\alpha}}$ stands for the Semiclassical Sobolev norm $\|\psi\|_{H^m_{1/\alpha}} := \sum_{\xi \in 2\pi\mathbb{Z}} \left| \left\langle \frac{1}{\alpha}\xi \right\rangle^m \hat{\psi}(\xi) \right|^2$. See figure 2.1.

As explained in detail in [7], from this and the decomposition (1.5) one can deduce the mixing property of $\widehat{E_\tau}$. We refer to the notations defined in the introduction.

Corollary 3. *Exponential Mixing of $\left(\widehat{E_\tau}, S^1 \times \mathbb{G}\right)$ for $\mathbb{G} \equiv U(1)$, $SU(2)$. If the conclusion of Theorem 2 hold, than for any $\rho > \frac{1}{\sqrt{E_{\min}}}$ there exists a finite rank operator $\hat{k}$ such that, for any observables $\Phi, \Psi \in C^\infty(S^1 \times \mathbb{G})$*

$$\left(\widehat{F}_\tau^n \Psi; \Phi\right)_{L^2(\hat{\mu}_\tau)} = \left(\hat{k}^n \Psi; \Phi\right)_{L^2(\hat{\mu}_\tau)} + \mathcal{O}(\rho^n).$$

Let Π_0 be the projector on $\mathcal{D}_0$. If $\lambda = 1$ is the only eigenvalue of $\widehat{F}_\tau$ on the unit circle[2] than $\hat{k}$ admits a spectral decomposition $\hat{k} = |1\rangle \langle \mu| \otimes \Pi_0 + \hat{r}$, $r_s(\hat{r}) < 1$ with $|1\rangle \langle \mu| (\varphi)(x) := \langle \mu| \varphi\rangle_{L^2(S^1)}$ and μ the SRB measure of E. Thus

$$C_{\Psi,\Phi}(n) - \int \bar{\Psi} d\hat{\mu}_\tau \int \Phi d\hat{\mu}_\tau = (\hat{r}^n \Psi; \Phi)_{L^2(\hat{\mu}_\tau)} + \mathcal{O}(\rho^n).$$

thus $\left(\widehat{E_\tau}, S^1 \times \mathbb{G}\right)$ is exponentially mixing.

Proof. [7] subsection 2.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Definition 4. For any $\lambda > 0$ consider $\mathcal{O}_\lambda = \{m \in \mathbb{R} \,|\, r_m \leq \lambda\}$. The set of *all resonances* of $\hat{F}_\alpha$ can be defined as

$$\text{Res}\left(\hat{F}_\alpha\right) := \lim_{\lambda \to 0} \bigcap_{m \in \mathcal{O}_\lambda} \text{spect}\left(\hat{F}_\alpha|_{H^m(S^1) \otimes \mathcal{D}_\alpha}\right).$$

In the spirit of [24] and [13] we show:

Theorem 5. *Let $\mathbb{G}$ be either $U(1)$ or $SU(2)$ as in Theorem 2. For any $\epsilon > 0$ let $D_\epsilon^{\mathbb{C}}$ be the open disc in $\mathbb{C}$ of radius ϵ. Then as $\alpha \to \infty$,*

$$(2.4) \qquad\qquad \sharp\left\{Res\left(\hat{F}_\alpha\right) \cap \mathbb{C}\backslash D_\epsilon^{\mathbb{C}}\right\} = \mathcal{O}\left(|\alpha|^{\frac{1}{2}\dim(K_\mathbb{G})+0}\right)$$

where $\dim(K_\mathbb{G})$ is the upper Minkowski dimension (definition 19) of the trapped set $K_\mathbb{G}$ of the canonical map associated to $\hat{F}_\alpha$. For $\mathbb{G} = U(1)$ this map (4.1) is defined on T^*S^1 and $K_{U(1)}$ is a compact subset of dimension between 1 and 2. For $\mathbb{G} = SU(2)$ this map (4.8) is defined on $T^*S^1 \times S^2$ and $K_{SU(2)}$ is a compact subset of dimension between 3 and 4. This behaviour is tested numerically on figure 2.2.

Remark 6. If we replace the arbitrary observables Φ, Ψ of Corollary 3 by functions $\Phi_\alpha(x,g), \Psi_\alpha(x,g)$ decomposing only on $C^\infty \otimes \mathcal{D}_\alpha$ (at x fixed, eigenvalues of the Laplace operator on $\mathbb{G}$ of eigenvalue λ_α [26] p. 550) then, for any $\epsilon > 0$, there exists a finite rank operator $\hat{k}_\alpha$, s.t.

$$\left(\widehat{F}_\tau^n \Psi_\alpha; \Phi_\alpha\right)_{L^2(\hat{\mu}_\tau)} = \left(\hat{k}_\alpha^n \Psi_\alpha; \Phi_\alpha\right)_{L^2(\hat{\mu}_\tau)} + \mathcal{O}(\epsilon^n).$$

with an estimation on the rank given by Theorem 5, growing as α grows.

3. $\hbar-$Pseudo differential theory

Before giving the proofs of Theorem 2 and 5 we first recall some basic facts from semiclassical analysis. This will give us the opportunity to fix some notations but the reader familiar with this theory can very well skip this section. We refer to [5, 10, 16, 3].

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[2]which is always the case if τ is not a co-boundary: [29] Appendix A
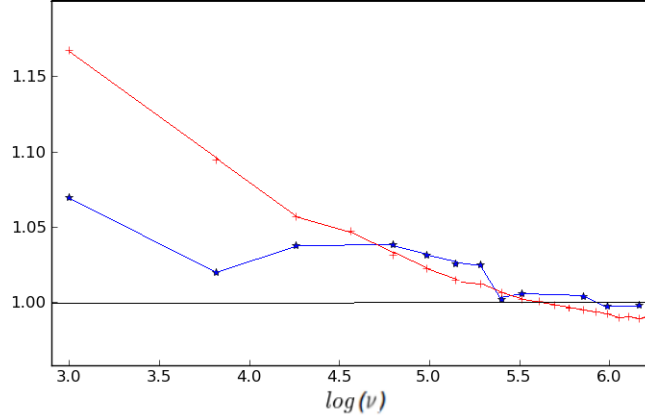
FIGURE 2.2. Here we restrict our attention to the Abelian skew extension of the linear map $E(x) = 2x$ with $\tau(x) = \frac{1}{2\pi} \cos(2\pi x) \in \mathbb{R}/\mathbb{Z}$. The +signs represent $\log(N_\nu)/\log(\nu)$ with N_ν the number of resonances of $\hat{F}_\nu$ larger in modulus than some fixed $\epsilon > 0$. The stars represent $1 + \log Vol(K_\nu)/\log(\nu)$ with K_ν a numerical approximation of the volume of a $\nu^{-\frac{1}{2}}$−neighbourhood of the associated trapped set K for $20 \leq \nu \leq 600$. If Theorem 5 where sharp, then it would imply that both quantities should converge at same speed to $\frac{1}{2} \dim K \sim 1$ (see Lemma 18). Thus numerics suggest that it is the case.

The symplectic structure of $\left(\mathbb{R}^{2d}, \omega = dx \wedge d\xi\right)$ induces a Poisson algebra on the set of classical observables $C^\infty\left(\mathbb{R}^{2d}; \mathbb{R}\right)$ if one defines the Poisson bracket of two such functions as $\{f, g\} := \omega\left(X_f, X_g\right)$ with X_f the Hamiltonian vector field generated by f. PDO theory stems from an attempt to transpose such an algebra to the set of formally self-adjoint operators acting on $L^2\left(\mathbb{R}^d\right)$, the latter interpreted as the quantum Hilbert space. To some observable f we associate, $\forall \hbar > 0$, and $\varphi \in \mathcal{S}$, its Weyl quantization:

$$(3.1) \qquad \mathrm{Op}_\hbar^w(f)\varphi(x) = \frac{1}{(2\pi\hbar)^d} \int e^{\frac{i}{\hbar}\xi(x-y)} f\left(\frac{x-y}{2}, \xi\right) \varphi(y)dy d\xi$$

f is than said to be the Weyl symbol of $\mathrm{Op}_\hbar^w(f) : \mathcal{S} \to \mathcal{S}'$. This expression makes sense if f is not real but if it is the operator is formally self-adjoint. If one restricts to the following class of symbols, given $m \in \mathbb{R}$ and $0 \leq \mu < \frac{1}{2}$

$$(3.2) \qquad S_\mu^m := \left\{ a_\hbar \in C^\infty\left(\mathbb{R}^{2d}\right) \mid \left|\partial_x^\alpha \partial_\xi^\beta a_\hbar\right| \leq C_{\alpha\beta}\hbar^{-\mu(|\alpha|+|\beta|)} \langle\xi\rangle^{m-|\beta|} \right\}$$

then, for any $a_\hbar \in S_\mu^m$, $\mathrm{Op}_\hbar^w(a_\hbar)$ maps $\mathcal{S}$ to $\mathcal{S}$ and extends continuously to a map $\mathcal{S}' \to \mathcal{S}'$. We write OPS_μ^m for the set of operators associated to S_μ^m. This class allows the construction of the Weyl quantization over smooth compact manifolds, with (3.1) taken in a local sense [27]. Symbols are then well defined as functions of

the cotangent bundle. In our case the manifold is the unit circle[3] and its cotangent bundle the cylinder T^*S^1. In the following we drop the subscript $\hbar$ for symbols in S_μ^m even-though they generally are one-parameter families of functions.

Lemma 7. *(L^2-continuity theorem 5.1 in [5]). If $a \in S_\mu^0$ than for any $\hbar > 0$, $\mathrm{Op}_\hbar^w(a)$ extends to a continuous operator on L^2. As $\hbar$ goes to zero, $\|\mathrm{Op}_\hbar^w(a)\|_{L^2 \to L^2} \leq$ $\sup |a| + \mathcal{O}(\hbar^{1-2\mu})$.*

The set of $\hbar$-PDO defined through the weyl quantization is an algebra and defines a star-algebra on the set of symbols, which coincides at first order with the Poisson algebra of $C^\infty\left(\mathbb{R}^{2d}\right)$ induced by $\omega = dx \wedge d\xi$:

Lemma 8. *(Composition [10] p. 109). Let a, $b \in S_\mu^{m_1}$, $S_\mu^{m_2}$. Than $\mathrm{Op}_\hbar^w(a)\mathrm{Op}_\hbar^w(b) \in OPS_\mu^{m_1+m_2}$, thus defining the star product $a \natural b$ such that $\mathrm{Op}_\hbar^w(a)\mathrm{Op}_\hbar^w(b) = \mathrm{Op}_\hbar^w(a \natural b)$. $a \natural b = ab \mod \hbar^{1-2\mu} S_\mu^{m_1+m_2}$ and Furthermore*

$$[\mathrm{Op}_\hbar^w(a), \mathrm{Op}_\hbar^w(b)] = \frac{\hbar}{i}\mathrm{Op}_\hbar^w(\{a, b\}) \mod \hbar^{2(1-2\mu)} S_\mu^{m_1+m_2-2}$$

The major consequence of this fact is that if U_t is one parameter-group of unitary operators satisfying the Shrödinger equation $-i\hbar\partial_t U_t := \mathrm{Op}_\hbar^w(H)U_t$, with H a real bounded symbol, than the classical and "quantum" dynamics are related at first order by the celebrated Egorov theorem:

Lemma 9. *(Egorov. section 9.2 in [5]). For any $a \in S_\mu^m$, $U_{-t}\mathrm{Op}_\hbar^w(a)U_t \in OPS_\mu^m$ and its symbol is $a \circ e^{tX_H} \mod \hbar^{1-2\mu} S_\mu^{m-1}$, with e^{tX_H} the time-t flow generated by the Hamiltonian vector field X_H.*

The operators U_t are a special kind of so called $\hbar-$Fourier-Integral-Operator (FIO) [16]. FIOs are always associated to a symplectic map on the classical phase space. Another important example of FIOs is given by pull-back operators U_κ : $\varphi \mapsto \varphi \circ \kappa$ with κ a smooth diffeomorphism of $\mathbb{R}^d$. The above Egorov theorem holds replacing the Hamiltonian flow by the canonical lift $\widetilde{\kappa}$ of κ on $\mathbb{R}^{2d}$ (seen as the cotangent bundle of $\mathbb{R}^d$),

$$\widetilde{\kappa} : (x, \xi) \mapsto (\kappa^{-1}(x), {}^t D_{\kappa^{-1}(x)}\kappa \cdot \xi).$$

4. CANONICAL MAPS

In this section we derive the canonical maps that play a central role in our approach.

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[3]In our context, since S^1 is an affine manifold it is not necessary to restrict to such a class. But since our techniques allow it, and since S^m is standard we will not work on the most general class allowed.

4.1. **The Abelian case.** When $\mathbb{G} \equiv U(1)$, the operator $\hat{F}_\nu$ reads $e^{i\nu\Omega}\mathcal{F}_E$ with $\mathcal{F}_E : \varphi \mapsto \varphi \circ E$ and $\Omega \in C^\infty(S^1)$. $\hat{F}_\nu$ can be seen as an $\hbar$−Fourier-Integral-Operator (FIO), with semiclassical parameter ν^{-1}. As explained in the previous section, the pull back operator $\mathcal{F}_E$ is one of the simplest examples of an FIO and is associated to the k−valued canonical lift $\widetilde{E}$ of E on the cotangent space T^*S^1 (recall that E_ϵ^{-1} are the inverse branches of E):

$$\widetilde{E}_\epsilon : (x,\xi) \mapsto (E_\epsilon^{-1}(x), E'\left(E_\epsilon^{-1}(x)\right)\xi) \; \epsilon = 0,...,k-1.$$

On the other hand the multiplication operator $e^{i\nu\Omega}$, $\nu \to \infty$ is also a very simple FIO and is associated to the time 1 flow generated by the Hamiltonian $\Omega(x)$, $(x,\xi) \mapsto (x,\xi + \Omega'(x))$. The canonical map associated to $\hat{F}_\nu$ is thus k−valued and reads

$$(4.1) \qquad F_\epsilon : (x,\xi) \mapsto (E_\epsilon^{-1}(x), E'\left(E_\epsilon^{-1}(x)\right)\xi + \Omega'\left(E_\epsilon^{-1}(x)\right)) \; \epsilon = 0,...,k-1.$$

The intuitive idea behind these maps is that wave packets localized both in direct and Fourier space near some point $(x,\nu\xi) =: (x,\xi_\nu)$ will be transformed, up to negligible terms as $\nu \to \infty$, in other wave packets localized near $F_\epsilon(x,\xi_\nu)$, $\epsilon = 0,...,k-1$. To the reader not familiar with semiclassical analysis we recommend the reading of section 3.2 in [7] where this simple idea is explained in detail.

Using the fact that $E_\epsilon^{-1} \circ E = Id_{S^1}$ and (2.1) the Egorov theorem of section 3 can be quoted with $\hat{F}_\nu$ in the role of the FIO as

Lemma 10. *(Egorov in the Abelian setting). Let $\hbar = \nu^{-1}; \nu > 0$. For any $a \in S_\mu^m(T^*S^1)$ any $\hbar > 0$, $\hat{F}_\nu^* Op_\hbar^w(a)\hat{F}_\nu \in OPS_\mu^m$ and its symbol reads*

$$(4.2) \qquad \sum_{\epsilon=0,...,k-1} \frac{a \circ F_\epsilon}{E' \circ E_\epsilon^{-1}} \; mod \; \hbar^{1-2\mu}S_\mu^{m-1}.$$

4.2. **The simplest non-Abelian case.** When $\mathbb{G} \equiv SU(2)$, the operator $\hat{F}_j$ reads $\hat{\tau}_j\left(\mathcal{F}_E \otimes \mathbb{I}_{\mathcal{D}_j}\right)$. We thus need to understand how the unitary operator $\hat{\tau}_j$ defined by $(\hat{\tau}_j\boldsymbol{\varphi})(x) := \hat{\tau}_j(x)\boldsymbol{\varphi}(x)$ could be an FIO and on which space its canonical map would act. In other words we have to define wave packets on $C^\infty(S^1) \otimes \mathcal{D}_j$ and exhibit the action of $\hat{\tau}_j$ upon them. For this purpose we shall use coherent states theory for compact Lie groups [20].

The Lie algebra of $SU(2)$ is a 3 dimensional real vector space spanned by three generators $iJ_l; l = 1,2,3$. The representations spaces $\mathcal{D}_j$ are eigenspaces of the Casimir operator $J^2 := \sum_{l=1}^3 J_l^2$ with eigenvalue $j(j+1)$, and an o.n. basis in $\mathcal{D}_j$ is given by the eigenvectors of J_3. In particular if one defines $J_\pm = J_1 \pm iJ_2$ there exists a unique normalized vector $|0\rangle \in \mathcal{D}_j$ such that $J_3|0\rangle = -j|0\rangle$ and $J_-|0\rangle = 0$, called the maximal weight vector and $\mathcal{D}_j = span\left\{J_+^k|0\rangle, k = 0,...,2j\right\}$. Let $\pi : \mathcal{D}_j \to \mathbb{P}(\mathcal{D}_j)$ be the canonical mapping on the projective space. Define the following subset of $\mathbb{P}(\mathcal{D}_j)$

$$(4.3) \qquad\qquad X_j := \left\{\pi\left(\hat{g}_j|0\rangle\right); g \in \mathbb{G}\right\},$$

i.e. the set of complex lines trough 0 in $\mathcal{D}_j$ spanned by the orbit of $|0\rangle$. The isotropy subgroup of $\pi(|0\rangle)$ is the set of elements $\{e^{i\theta J_3}\}$, a U(1) subgroup of $\mathbb{G}$. Thus X_j is isomorphic to the coset space $SU(2)/U(1) \cong S^2$. The isomorphism $\phi : X_j \to S^2$ can be fixed once and for all by setting $\phi(\pi(\hat{g}_j |0\rangle)) = R_g \mathbf{n}_0$ with $\mathbf{n}_0$ the south pole of the sphere and $R_g \in SO(3)$ the rotation corresponding to the adjoint representation of g in $\mathbb{R}^3$. A point $\mathbf{n}$ on the sphere is thus associated to an orthogonal projector $|\mathbf{n}\rangle \langle\mathbf{n}|$ on $\mathcal{D}_j$ with $|\mathbf{n}\rangle$ being any $\hat{g}_j |0\rangle$ s.t. $[g] \equiv \mathbf{n}$. This mapping is called the *quantization of the sphere* and the vectors $|\mathbf{n}\rangle$ the *coherent states*. Furthermore, since $\int_{S^2} |\mathbf{n}\rangle \langle\mathbf{n}| \, d\mathbf{n}$ commutes with all elements of $\mathbb{G}$, by Shur's lemma it is a multiple of the identity $\mathbb{I}_{\mathcal{D}_j}$. An algebraic calculation gives ([20] p. 63)

$$(4.4) \qquad \frac{\dim \mathcal{D}_j}{4\pi} \int_{S^2} |\mathbf{n}\rangle \langle\mathbf{n}| \, d\mathbf{n} = \mathbb{I}_{\mathcal{D}_j}.$$

Also one can show that the coherent sates are localized, as $j \to \infty$, on the sphere: $|\langle\mathbf{n}'| \, \mathbf{n}\rangle|^2 = \left|\frac{1+\mathbf{n}'\cdot\mathbf{n}}{2}\right|^{2j}$. A natural way to quantize smooth observables is to put

$$(4.5) \qquad \mathrm{Op}_j^{AW} : \begin{array}{ccc} C^\infty(S^2) & \to & \mathrm{End}(\mathcal{D}_j) \\ a(\mathbf{n}) & \mapsto & \int_{S^2} a(\mathbf{n}) |\mathbf{n}\rangle \langle\mathbf{n}| \, d\mathbf{n}_j \end{array} \qquad d\mathbf{n}_j := \frac{\dim \mathcal{D}_j}{4\pi}.$$

This mapping is surjective. In fact, because we have chosen a maximal weigh vector as a starting point, X_j has a Kaehlerian structure inherited from the projective space ([12] p. 168). The symplectic form on X_j reads $\omega_j = j\omega_{S^2}$ with ω_{S^2} the canonical symplectic form on S^2 and (4.5) corresponds to the geometric quantization of (S^2, ω_{S^2}) with $\mathcal{D}_j$ as the quantum Hilbert space [30]; the following is a special case of a more general result:

Theorem 11. *(Cahen-Gutt-Rawnsley [17]). For any $a, b \in C^\infty(S^1)$*

$$\mathrm{Op}_j^{AW}(a)\mathrm{Op}_j^{AW}(b) = \mathrm{Op}_j^{AW}(a\natural b),$$

with $a\natural b = ab + O(j^{-1})$. Furthermore

$$-\frac{i}{j}\left[\mathrm{Op}_j^{AW}(a), \mathrm{Op}_j^{AW}(b)\right] = \mathrm{Op}_j^{AW}(\{a, b\}) + \mathcal{O}_{End(\mathcal{D}_j)}(j^{-1}).$$

Remark 12. The representations of group elements are natural FIOs in this setting as one can check readily that, for any observable a, and $g \in \mathbb{G}$,

$$\hat{g}_j^{-1}\mathrm{Op}_j^{AW}(a)\hat{g}_j = \mathrm{Op}_j^{AW}(a \circ R_g).$$

Since $\hat{g}_j = e^{iju\cdot\frac{J}{j}}$ for some $u \in \mathbb{R}^3$ this means that $u \cdot \frac{J}{j} = \mathrm{Op}_j^{AW}(u \cdot \mathbf{n} + \mathcal{O}(j^{-1}))$ so that the Hamiltonian time-1 flow on S^2 coincides with the rotation $R_g \in SO(3)$.

We can now define a quantization of the full symplectic phase space

$$(4.6) \qquad \left(T^*S^1 \times S^2; dx \wedge d\xi + \omega_{S^2}\right)$$

To any classical observable $a \in C^\infty(T^*S^1 \times S^2)$ we associate an operator $\mathrm{Op}_j(a)$: $\mathcal{S}(S^1) \otimes \mathcal{D}_j \to \mathcal{S}'(S^1) \otimes \mathcal{D}_j$ defined by

$$(4.7) \qquad \mathrm{Op}_j(a) := \mathrm{Op}_j^{AW} \circ \mathrm{Op}_{j^{-1}}^w(a)$$

with $\mathrm{Op}_\hbar^w; \hbar > 0$ the usual Weyl quantization (3.1) defined on the circle. By composition the quantization (4.7) obeys the correspondence principle: The composition of two such PDO is a PDO whose principal term is the product of the symbols and the principal term of their commutator is $(-i/j)$ times the Poisson bracket of the symbols. Since SU(2) is simply connected we can write $\tau(x)$ as $e^{i\Omega(x)\cdot J}$ with $\Omega \in C^\infty(S^1; \mathbb{R}^3)$ a smooth vector valued function. Using remark 12, we have that

$$\hat{\tau}_j = \exp\left(ij\mathrm{Op}_j(a)\right); \text{ with } a(x,\mathbf{n}) = \Omega(x) \cdot \mathbf{n} + \mathcal{O}(j^{-1}).$$

This is precisely the formal expression of an FIO associated to the time-1 flow generated by the Hamiltonian vector field associated to $\Omega(x) \cdot \mathbf{n}$:

$$\dot{x} = 0; \ \dot{\xi} = -\Omega'(x) \cdot \mathbf{n}; \ \dot{\mathbf{n}} = \Omega(x) \wedge \mathbf{n}$$

After integration the time-1 flow reads:

$$(x, \xi, \mathbf{n}) \mapsto (x, \ \xi + \mathbf{n} \cdot H(x, \mathbf{n}), \ R_{\tau(x)}\mathbf{n}),$$

with $H(x,\mathbf{n}) := \mathbf{n} \cdot \int_0^1 \tilde{R}_{t\Omega(x)}\Omega'(x)dt$, and $\tilde{R}_u$ the rotation in $\mathbb{R}^3$ of axis u. Coming back to the operator $\hat{F}_j = \hat{\tau}_j\left(\mathcal{F}_E \otimes \mathbb{I}_{\mathcal{D}_j}\right)$ we see now that it is indeed an FIO and by composition its k-valued canonical map reads, with $\epsilon = 0, ..., k-1$:

$$(4.8) \qquad F_\epsilon : (x, \xi, \mathbf{n}) \mapsto \left(E_\epsilon^{-1}(x), E'\left(E_\epsilon^{-1}(x)\right)\xi + H(E_\epsilon^{-1}(x); \mathbf{n}), R_{\tau\left(E_\epsilon^{-1}(x)\right)}\mathbf{n}\right).$$

Using the Campbell-Hausdorff formula ([26] p. 541) one can show that[4]

$$H(x, \mathbf{n}) = -\frac{i}{j}\left\langle \mathbf{n}\right| \hat{\tau}_j^{-1}\frac{d\hat{\tau}_j}{dx}(x)\left|\mathbf{n}\right\rangle,$$

the Wick symbol [17] of $-ij^{-1}\left(\hat{\tau}_j^{-1}\frac{d}{dx}\hat{\tau}_j\right)(x) \in \mathfrak{su}(2)$.

---

[4]This result is not surprising if one considers the transport by $\hat{\tau}_j$ of generalized wave packets $\varphi_{x,\xi,\mathbf{n}} := \varphi_{x,\xi} \otimes |\mathbf{n}\rangle$ with $\varphi_{x,\xi}$ a Gaussian wave-packet of width $j^{-1/2}$ as defined in [7] section 3.2. By the localization property of both the coherent states and the Gaussian wave packets

$$\left(\varphi_{y,\eta,\mathbf{n}'}|\hat{\tau}_j\varphi_{x,\xi,\mathbf{n}}\right)_{L^2(S^1)\otimes\mathcal{D}_j} := \int_{S^1} \overline{\varphi_{y,\eta}(z)}\varphi_{x,\xi}(z)\left\langle \mathbf{n}'|\hat{\tau}_j(z)\mathbf{n}\right\rangle dz$$

will be negligible as j grows if $y \neq x$ and if $y = x$ negligible if $\mathbf{n}' \neq R_{\tau(x)}\mathbf{n}$. On the other hand, for some $c > 0$

$$\left|\left(\varphi_{x,\eta,R_{\tau(x)}\mathbf{n}}|\hat{\tau}_j\varphi_{x,\xi,\mathbf{n}}\right)\right| := \left|\int_{S^1} e^{-j(x-z)^2}e^{ij(\xi-\eta)}\left\langle \mathbf{n}|\hat{\tau}_j(x)^{-1}\hat{\tau}_j(z)\mathbf{n}\right\rangle dz\right| + \mathcal{O}(e^{-cj}).$$

$\left\langle \mathbf{n}|\hat{\tau}_j(x)^{-1}\hat{\tau}_j(z)\mathbf{n}\right\rangle$ is maximal and equal to one for $z = x$. If we write this term as $\rho(z)e^{-ij\psi(z)}$ then the stationary phase theorem states that the above expression will be negligible whenever $\xi - \eta - \frac{i}{j}\left\langle \mathbf{n}|\hat{\tau}_j(x)^{-1}\frac{d}{dx}\hat{\tau}_j(x)\mathbf{n}\right\rangle \neq 0$.

Set $\hbar \equiv j^{-1}$, $j > 0$ to define the class, as (3.2), given $m \in \mathbb{R}$ and $0 \leq \mu < \frac{1}{2}$:

$$(4.9) \quad S_\mu^m \left(T^* S^1 \times S^2 \right) := \left\{ a_\hbar \in C^\infty \,;\, |\partial_x^\alpha \partial_\xi^\beta \partial_{\mathbf{n}}^\gamma a_\hbar| \leq C_{\alpha\beta} \hbar^{-\mu(\alpha+\beta+|\gamma|)} \langle \xi \rangle^{m-|\beta|} \right\}.$$

In this setting the Egorov theorem can then be stated as

Lemma 13. *(Egorov in the non-Abelian setting). Let $a \in S_\mu^m \left(T^* S^1 \times S^2 \right)$, then $\hat{F}_j^* Op_j \left(a \right) \hat{F}_j \in OPS_\mu^m$ and its principal symbol reads*

$$(4.10) \qquad \sum_{\epsilon=0,\ldots,k-1} \frac{a \circ F_\epsilon}{E' \circ E_\epsilon^{-1}} \ \ mod \ \hbar^{1-2\mu} S_\mu^{m-1}.$$

5. Dynamics on Phase space

In this section we derive some elementary facts about the canonical maps (4.1) and (4.8) associated respectively to $\left\{ \hat{F}_\nu \right\}_{\nu \in \mathbb{Z}}$ and $\left\{ \hat{F}_j \right\}_{j \in \frac{1}{2}\mathbb{N}}$. We introduce the compact trapped sets of Theorem 5 and give the hypothesis on which Theorem 2 is based.

5.1. **Time$-n$ dynamics.** Let $\mathcal{A} := \{0, \ldots, k-1\}$ be the alphabet and

$$\mathcal{A}^n := \left\{ \epsilon = \epsilon_1 \epsilon_2 \ldots \epsilon_2; \epsilon_i \in \mathcal{A} \right\},$$

the set of k^n words on length $n > 0$ written with $\mathcal{A}$. For any $\epsilon \in \mathcal{A}^n$ define $E_\epsilon^{-n} := E_{\epsilon_n}^{-1} \circ \ldots \circ E_{\epsilon_1}^{-1}$ and put $x_\epsilon := E_\epsilon^{-n}(x)$. The expansion rate of this trajectory is then

$$E_\epsilon'(x) := (E^n)' \left(x_\epsilon \right) = \prod_{j=1}^n E'(x_{\epsilon|_j})$$

with $\epsilon|_j := \epsilon_1 \ldots \epsilon_j$ the troncation at the $j-$th letter of the word ϵ. We put $\forall \epsilon \in \mathcal{A}^n$:

$$\xi_{x,\epsilon} := E_\epsilon'(x)\{\xi - S_\epsilon^{U(1)}(x)\}$$

with

$$(5.1) \qquad S_\epsilon^{U(1)}(x) := - \sum_{j=1}^n E_{\epsilon|_j}'(x)^{-1} \cdot \Omega' \left(x_{\epsilon|_j} \right),$$

and Ω as in (4.1). In the same manner, if

$$\mathbf{n}_{x,\epsilon} := R_{\tau(x_\epsilon)} \circ R_{\tau(x_{\epsilon|_{n-1}})} \circ \ldots \circ R_{\tau(x_{\epsilon_1})} \mathbf{n}$$

define

$$\xi_{x,\mathbf{n},\epsilon} := E_\epsilon'(x)\{\xi - S_\epsilon^{SU(2)}(x, \mathbf{n})\}$$

with

$$(5.2) \qquad S_\epsilon^{SU(2)}(x, \mathbf{n}) := - \sum_{j=1}^n E_{\epsilon|_j}'(x)^{-1} \cdot H_\dagger \left(x_{\epsilon|_j}, \mathbf{n}_{x,\epsilon|_j} \right),$$

and $H_\dagger(x, \mathbf{n}) := H(x, R_{\tau(x)}^{-1}\mathbf{n})$, H as in (4.8). With these notations the time-n dynamics read

$$(5.3) \qquad F_\epsilon^n(x, \xi) = (x_\epsilon, \xi_{x,\epsilon}) \text{ and } F_\epsilon^n(x, \xi, \mathbf{n}) = (x_\epsilon, \xi_{x,\mathbf{n},\epsilon}, \mathbf{n}_{x,\epsilon}); \, \epsilon \in \mathcal{A}^n$$

for respectively the Abelian and non-Abelian case. The inverse maps F^{-n} are single valued and read,

$$(5.4) \qquad F^{-n}(x, \xi) = (\, E^n x, (E^n)'(x)^{-1}\{\xi - \sum_{j=0}^{n-1} (E^j)'(x) \cdot \Omega'(E^j x)\} \,)$$

in the Abelian setting, and if $\mathbf{n}_x^{(j)} := R_{\tau(E^{j-1}x)}^{-1} \circ ... \circ R_{\tau(x)}^{-1}\mathbf{n}$, in the non-Abelian setting:

$$(5.5) \quad F^{-n}(x, \xi, \mathbf{n}) = (\, E^n x, \, (E^n)'(x)^{-1}\{\xi - \sum_{j=0}^{n-1} (E^j)'(x) \cdot H_\dagger(E^j x, \mathbf{n}_x^{(j)})\}, \, \mathbf{n}_x^{(n)} \,)$$

5.2. **The trapped sets.** We refer to the notations defined in the previous subsection. A fundamental feature of the classical dynamics is that:

Lemma 14. *For any $1 < \kappa < e_{\min}$, there exists $R > 0$, s.t. for any $\epsilon \in \mathcal{A}$, $|\xi| \geq R \Rightarrow |\xi_{x,\epsilon}| \geq \kappa |\xi|$ and $|\xi_{x,\mathbf{n},\epsilon}| \geq \kappa |\xi|$.*

Proof. If $\xi > 0$, than $\xi_{x,\epsilon}$ (resp. $\xi_{x,\mathbf{n},\epsilon}$) will be larger than $\kappa\xi$ for some $1 < \kappa < e_{\min}$ iff $\kappa\xi \geq e_{\min}\xi - C_-^\mathbb{G} \Leftrightarrow \xi \geq C_-^\mathbb{G}/(e_{\min} - \kappa)$, with $C_-^{U(1)} = |\min_x \Omega'(x)|$ and $C_-^{SU(2)} = |\min_{x,\mathbf{n}} H(x, \mathbf{n})|$. On the other hand if $\xi < 0$ then $\xi_{x,\epsilon}$ (resp. $\xi_{x,\mathbf{n},\epsilon}$) will be smaller than $\kappa\xi$ iff $\kappa\xi \leq e_{\min}\xi + C_+^\mathbb{G} \Leftrightarrow \xi \leq -C_+^\mathbb{G}/(e_{\min} - \kappa)$, with $C_+^{U(1)} = |\max_x \Omega'(x)|$ and $C_+^{SU(2)} = |\max_{x,\mathbf{n}} H(x, \mathbf{n})|$. So with $R := \max_\mathbb{G} \max\{C_-^\mathbb{G}, C_+^\mathbb{G}\}/(e_{\min} - \kappa)$ the lemma holds. $\qquad \square$

As a consequence, for both maps (4.1) and (4.8) there exists a non-empty compact set of points from which some trajectories do not escape as $n \to \infty$. Indeed, taking $R > 0$ large enough and $Z_{U(1)} := S^1 \times [-R; R]$, $Z_{SU(2)} := S^1 \times [-R; R] \times S^2$, $K_\mathbb{G}$ can be defined as the limit of a sequence of nested non-empty compacts sets (see fig. 5.1):

$$(5.6) \qquad\qquad K_\mathbb{G} := \bigcap_{n \geq 0} F^{-n}(Z_\mathbb{G}).$$

For some word $\epsilon \in \mathcal{A}^n$ we define $\overline{\epsilon} := \epsilon\overline{00}$ the word of infinite length completed from ϵ with zeros. Put $\overline{\mathcal{A}^n} := \{\overline{\epsilon}; \epsilon \in \mathcal{A}^n\}$. One can than define the set of infinite words as $\mathcal{A}^\infty := \cup_{n>0}\overline{\mathcal{A}^n}$.

Lemma 15. *$K_\mathbb{G}$ can be seen as the closure of the union of graphs of smooth uniformly bounded functions $\mathsf{S}_\epsilon^\mathbb{G}; \epsilon \in \mathcal{A}^\infty$ over, resp. S^1 and $S^1 \times S^2$:*

$$(5.7) \qquad\qquad K_\mathbb{G} = \overline{\cup_{\epsilon \in \mathcal{A}^\infty} \mathcal{G}\mathsf{S}_\epsilon^\mathbb{G}}.$$
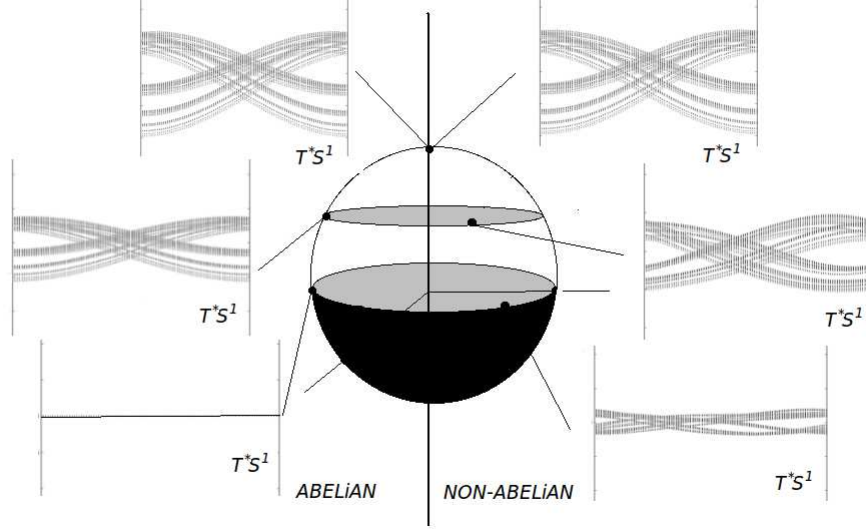
FIGURE 5.1. Numerical computation of $K_{SU(2)}$ corresponding to the skew extension of the linear map $E(x) = 2x \bmod 1$ for two particular expression of τ. On the left $\tau(x) = e^{i\cos(2\pi x)J_3}$ with J_3 the generator of the rotations around the vertical axis. In this case τ maps S^1 to a $U(1)$ subgroup of $SU(2)$. The induced dynamics on the sphere is not transitive as it leaves invariant the geodesics parallel to the equator. Over any of these, the trapped set corresponds to $K_{U(1)}$ for $\tau(x) = \frac{1}{2\pi}\mathbf{n}_3\cos(2\pi x)$, and degenerates above the equator (this is consistent with the fact that the canonical map ought not be partially captive in this case). On the right we break the degeneracy by taking $\tau(x) = e^{i\cos(2\pi x)J_3 + i0.2\cos(2\pi x)J_1}$.

Furthermore, if $S_{\max}^{\mathbb{G}}$ is the uniform bound of the sequence $\{|\mathsf{S}_\epsilon^{\mathbb{G}}|\}$ than any point in $F^{-n}(Z_{\mathbb{G}})$ is at distance at most $\left(R + S_{\max}^{\mathbb{G}}\right)e_{\min}^{-n}$ from $K_{\mathbb{G}}$.

Proof. Points in $F^{-n}\left(Z_{U(1)}\right)$ can be written as

$$\bigcap_{x\in S^1}\bigcap_{\epsilon\in\mathcal{A}^n}\left\{F^{-n}(x_\epsilon, \xi); \xi\in[-R; R]\right\}.$$

Using (5.4) a short calculation gives $F^{-n}(x_\epsilon, \xi) = \left(x, E_\epsilon'(x)^{-1}\cdot\xi + S_\epsilon^{U(1)}(x)\right)$ with $S_\epsilon^{U(1)}$ as in (5.1). As n grows we have that $|E_\epsilon'(x)^{-1}\cdot\xi| \leq e_{\min}^{-n}R \to 0$. On the other hand, for any $\epsilon\in\mathcal{A}^\infty$, any $n > 0$, $\left|S_{\epsilon|n}^{U(1)}(x)\right| \leq \|\Omega'\|_\infty\frac{1}{e_{\min}-1} =: S_{max}^{U(1)}$ and $\left|S_{\epsilon|n}^{U(1)}(x) - S_{\epsilon|n-1}^{U(1)}(x)\right| = \left|E_{\epsilon|n}'(x)^{-1}\cdot\Omega'\left(x_{\epsilon|n}\right)\right| \leq e_{\min}^{-n}\|\Omega'\|_\infty$. The same kind of estimates hold true for the sequences $\partial_x^\alpha S_{\epsilon|n}^{U(1)}(x)$ for any order of derivation proving the convergence in $C^\infty(S^1)$ of the sequence of functions $\left(S_{\epsilon|n}^{U(1)}(x)\right)_{n\geq 0}$.

Let us write $\mathsf{S}_\epsilon^{U(1)} = \lim_{n\to\infty} S_{\epsilon|n}^{U(1)}$ and $\mathcal{G}\mathsf{S}_\epsilon^{U(1)} := \left\{ (x, \mathsf{S}_\epsilon^{U(1)}(x)) \, ; x \in I \right\}$ its graph over S^1. As n grows $F^{-n}\left(Z_{U(1)}\right)$ converges to $\cup_{\bar\epsilon \in \overline{\mathcal{A}^n}} \mathcal{G}\mathsf{S}_{\bar\epsilon}^{U(1)}$ and this gives (5.7) for $\mathbb{G} \equiv U(1)$. Since, for any $\epsilon \in \mathcal{A}^\infty$,

$$\left| E'_{\epsilon|n}(x)^{-1} \cdot \xi + S_{\epsilon|n}^{U(1)}(x) - \mathsf{S}_\epsilon^{U(1)}(x) \right| \leq e_{\min}^{-n}\left(R + S_{max}^{U(1)}\right),$$

we get that points in $F^{-n}\left(Z_{U(1)}\right)$ lie at distance at most $e_{\min}^{-n}\left(R + S_{max}^{U(1)}\right)$ from $K_{U(1)}$. The exact same argument can be carried out for $\mathbb{G} \equiv SU(2)$ with $S_{\epsilon|n}^{SU(2)}$ as in (5.2) which is uniformly bounded by $\|H\|_\infty \frac{1}{e_{\min}-1} =: S_{max}^{SU(2)}$, and converges in the $C^\infty-$topology to $\mathsf{S}_\epsilon^{SU(2)}$. $\square$

We do not know much more at present about these elusive sets [28]. Using the characterization (5.7) one can show quite easily that over some point $\{x\}$ or $\{(x, \mathbf{n})\}$ the trapped set is either reduced to a unique point $\{\xi\}$ or has no isolated points. This however does not inform us about their dimension more than the obvious bound $1 \leq \dim K_{U(1)} \leq 2$ and $3 \leq \dim K_{SU(2)} \leq 4$.

5.3. **The partially captive property.** For any starting point on $K_\mathbb{G}$, for any time $n > 0$, at least one of k^n different trajectories Stays in $K_\mathbb{G}$. In the light of lemma 15 we have obvious trapped trajectories from the fact that $\forall \epsilon \in \mathcal{A}^\infty, \epsilon_0 \in \mathcal{A}$,

$$F_{\epsilon_0}\left(x, \mathsf{S}_{\epsilon_0\epsilon}^{U(1)}(x)\right) = \left(x_{\epsilon_0}, \mathsf{S}_\epsilon^{U(1)}(x_{\epsilon_0})\right)$$

and similarly

$$F_{\epsilon_0}\left(x, \mathsf{S}_{\epsilon_0\epsilon}^{SU(2)}(x, \mathbf{n}), \mathbf{n}\right) = \left(x_{\epsilon_0}, \mathsf{S}_\epsilon^{SU(2)}(x_{\epsilon_0}, \mathbf{n}_{x,\epsilon_0}), \mathbf{n}_{x,\epsilon_0}\right),$$

showing that points in $K_\mathbb{G}$ lying on the branch $\mathcal{G}\mathsf{S}_{\epsilon_0\epsilon}^\mathbb{G}$ jump to the branch $\mathcal{G}\mathsf{S}_\epsilon^\mathbb{G}$ and so on. When τ is not a co-boundary, its seems only an unlikely coincidence that some other trajectory $\epsilon' \neq \epsilon_0$ would yield $\mathsf{S}_{\epsilon_0\epsilon}^{U(1)}(x) = \mathsf{S}_{\tilde\epsilon}^{U(1)}(x_{\epsilon'})$ or $\mathsf{S}_{\epsilon_0\epsilon}^{SU(2)}(x, \mathbf{n}) = \mathsf{S}_{\tilde\epsilon}^{SU(2)}(x_{\epsilon'}, \mathbf{n}_{x,\epsilon'})$, for some $\tilde\epsilon \in \mathcal{A}^\infty$. Thus one expects most trajectories to eventually escape in the non-compact direction.

Definition 16. Let $\mathcal{N}(n) \leq k^n$ be the maximal cardinality (over different starting points in $Z_\mathbb{G}$) of the set of trajectories $\epsilon \in \mathcal{A}^n$ that *do not escape* from $Z_\mathbb{G}$. The map $\widehat{E}_\tau$ of eq.(1.1) will be called *partially captive* iff:

$$(5.8) \qquad\qquad \lim_{n\to\infty} \frac{\log\left(\mathcal{N}(n)\right)}{n} = 0.$$

This is the hypothesis under which Theorem 2 holds. It is known to be generically true in the Abelian setting [29], but one can reasonably expect the arguments of the quoted article to remain valid in the non-Abelian case.

6. Proof of theorem 2

The Abelian case is treated in detail in [7]. We simply adapt Faure's proof to this context. In the following $1 < \kappa < e_{\min}$ and $R > 0$ are chosen as in lemma 14. Using the quantization rule (4.7), for any $j > 0$, $m < 0$ consider the following Hilbert spaces of distributions

$$H_{j-1}^m \left(S^1 \right) \otimes \mathcal{D}_j = \mathrm{Op}_j \left(A_m \right)^{-1} \left(L^2 \otimes \mathcal{D}_j \right),$$

where $A_m(x, \xi, \mathbf{n}) \equiv A_m(|\xi|) \in (0, 1]$ is an elliptic symbol in $S_0^m \left(T^* S^1 \times S^2 \right)$ (independent of $x, \mathbf{n}$), constant and equal to one for $|\xi| \leq R$ and equal to $R^{|m|} |\xi|^m$ pour $|\xi| \geq R + \eta$ with $\eta > 0$ arbitrary small. As subspaces of $\mathcal{D}'(S^1)$, $H_{j-1}^m \left(S^1 \right)$ and $H^m \left(S^1 \right)$ are isomorphic to one-an-other but posses different norms. Since the spectrum does not depend on the choice of a norm, the spectrum of $\hat{F}_j : H_{j-1}^m \left(S^1 \right) \otimes \mathcal{D}_j \to H_{j-1}^m \left(S^1 \right) \otimes \mathcal{D}_j$ is no other than the Ruelle spectrum of resonances introduced in theorem 1. Consider

$$\hat{Q}_m := \mathrm{Op}_j \left(A_m \right) \hat{F}_j \mathrm{Op}_j \left(A_m \right)^{-1}.$$

By construction $\hat{Q}_m$ acts in $L^2 \left(S^1 \right) \otimes \mathcal{D}_j$ and is unitary equivalent to $\hat{F}_j|_{H_{j-1}^m \otimes \mathcal{D}_j}$. On the other hand [14] $\forall n \in \mathbb{N}^*$

$$r_s \left(\hat{Q}_m \right) \leq \left\| \hat{Q}_m^n \right\|^{\frac{1}{n}} \leq \left\| \hat{Q}_m^{n*} \hat{Q}_m^n \right\|^{\frac{1}{2n}}.$$

Define

$$\hat{P}^{(n)} := \hat{Q}_m^{n*} \hat{Q}^n = \mathrm{Op}_j \left(A_m \right)^{-1} \hat{F}_j^{n*} \mathrm{Op}_j \left(A_m^n \right) \hat{F}_j^n \mathrm{Op}_j \left(A_m \right)^{-1}.$$

From Egorov (4.10) and composition theorems, using the notations of subsection 5.1, we get that $\hat{P}^{(n)} \in OPS_0^m$ and its symbol reads

$$P^{(n)} \left(x, \xi; \mathbf{n} \right) = \sum_{\epsilon \in \mathcal{A}^n} \frac{1}{E'_\epsilon(x)} \frac{A_m^2(\xi_{x, \mathbf{n}, \epsilon})}{A_m^2(\xi)} \mod j^{-1} S_0^{-1}.$$

Faure's simple idea is to use the basic properties of the classical dynamics to bound this positive symbol, and then use the L^2-continuity theorem (lemma 7) to conclude. At $x \in S^1$ and $\mathbf{n} \in S^2$ fixed we distinguish three cases. Using lemma 14 we get:

(1) If $|\xi| > R$, then $\forall \epsilon \in \mathcal{A}^n$, $\frac{A_m^2(\xi_{x, \mathbf{n}, \epsilon})}{A_m^2(\xi)} \leq \left(\kappa^{2m} \right)^n$.

(2) Si $|\xi| \leq R$ but $|\xi_{x, \mathbf{n}, \epsilon|_{n-1}}| > R$ then we can write

$$\frac{A_m^2(\xi_{x, \mathbf{n}, \epsilon})}{A_m^2(\xi)} = \overbrace{\frac{A_m^2(\xi_{x, \mathbf{n}, \epsilon})}{A_m^2(\xi_{x, \mathbf{n}, \epsilon|_{n-1}})}}^{\leq \kappa^{2m}} \overbrace{\frac{A_m^2(\xi_{x, \mathbf{n}, \epsilon|_{n-1}})}{A_m^2(\xi_{x, \mathbf{n}, \epsilon|_{n-2}})} ... \frac{A_m^2(\xi_{x, \mathbf{n}, \epsilon_1})}{A_m^2(\xi)}}^{\leq 1} \leq \kappa^{2m}.$$

(3) In all other case ($|\xi| \leq R|$ and $|\xi_{x, \mathbf{n}, \epsilon|_{n-1}}| \leq R$), $\frac{A_m^2(\xi_{x, \mathbf{n}, \epsilon})}{A_m^2(\xi)} \leq 1$, but by definition 16, the number of such trajectories is bounded by $\mathcal{N}(n-1)$.

Using this decomposition we get

$$P^{(n)}(x, \xi; \mathbf{n}) \leq \frac{1}{E_{\min}^n}\left(\left(k^n - \mathcal{N}(n-1)\right)\kappa^{2m} + \mathcal{N}(n-1)\right) + \mathcal{O}_n\left(j^{-1}\right)$$

Set $\mathcal{B}(n) := \left(\frac{k}{E_{\min}}\right)^n \kappa^{2m} + \frac{\mathcal{N}(n-1)}{E_{\min}^n}$. Remark that at n fixed the first term goes to zero as $m \to -\infty$. The L^2-continuity theorem gives

$$\left\|\hat{P}^{(n)}\right\| \leq \mathcal{B}(n) + \mathcal{O}_n\left(j^{-1}\right),$$

so

$$r_s\left(\hat{Q}_m\right) \leq \left(\mathcal{B}(n) + \mathcal{O}_n\left(j^{-1}\right)\right)^{\frac{1}{2n}} ; \ \forall n \in \mathbb{N}^*.$$

Letting first $j \to \infty$, then $m \to -\infty$ et finally $n \to \infty$ we finally obtain a nice expression for the the spectral radius, as j grows:

$$r_s\left(\hat{F}_j|_{H^m \otimes \mathcal{D}_j}\right) \leq \sqrt{\frac{1}{E_{\min}}\exp\left(\liminf_{n\to\infty}\left(\frac{\log\left(\mathcal{N}(n)\right)}{n}\right)\right)} + o(1).$$

The partially captive assumption (5.8) then yields (2.3) of Theorem 2. The second statement of Theorem 2 is derived using $\left\|\hat{Q}_m^n\right\| \leq \sqrt{\left\|\hat{P}^{(n)}\right\|} = (\mathcal{B}(n) + \mathcal{O}_n(j^{-1}))^{\frac{1}{2}}$ (polar decomposition [11]). With the partially captive assumption, for any $c > 0$ an n large enough $\mathcal{N}(n) < e^{nc}$, so for any $\rho > E_{\min}^{-1/2}$, $\frac{\mathcal{N}(n)}{E_{\min}^n} < \rho^{2n}$. Thus for m sufficiently negative, j, n sufficiently large $\left\|\hat{Q}_m^n\right\| := \left\|\hat{F}^n\right\|_{H_{j^{-1}}^m \otimes \mathcal{D}_j} \leq \rho^n$.

7. Proof of Theorem 5

To treat simultaneously both cases $\mathbb{G} \equiv U(1)$; $\mathbb{G} \equiv SU(2)$ let us fix some notations. Let $M_{\mathbb{G}}$ be the classical phase space associated to the FIO $\hat{F}_\alpha$; so $M_{U(1)} \equiv T^*S^1$ and $M_{SU(2)} \equiv T^*S^1 \times S^2$. We write points on $M_{\mathbb{G}}$ as $\rho = (\rho_c, \xi)$ with ρ_c the components along the compact directions of $M_{\mathbb{G}}$. We set $\hbar = \nu^{-1}$, $\nu > 0$ or $\hbar = j^{-1}$, $j > 0$ and Op will stand for the associated quantification either (3.1) or (4.7).

7.1. **The escape function.** For any closed subset A of $M_{\mathbb{G}}$ we denote by A^δ its closed $\delta-$neighbourhood.

Lemma 17. (*Existence of an escape function*). $\exists \mathsf{C}_0, \mathsf{C}_1 > 0$ *such that* $\forall m < 0$, $\forall 0 \leq \mu < \frac{1}{2}$; *and* $\forall 1 < \kappa < e_{\min}$, *there exists an elliptic symbol in* $\hbar^{\mu m} S_\mu^m(M_{\mathbb{G}})$, *called the* escape function *and written* $A_{m,\mu}^{\mathbb{G}}$, *satisfying the following property:*

$\forall \rho \notin K_{\mathbb{G}}^{\mathsf{C}_0 \hbar^\mu}$, $\forall \epsilon \in \mathcal{A}$, $A_{m,\mu}^{\mathbb{G}}$ *deacreases stricly along the trajectories of* F_ϵ:

$$(7.1) \qquad\qquad \frac{A_{m,\mu}^{\mathbb{G}} \circ F_\epsilon}{A_{m,\mu}^{\mathbb{G}}}(\rho) \leq \mathsf{C}_1 \kappa^m.$$

Proof. Let $m < 0$. Let R be as in lemma 14 and define $A_m \in S_0^m(M_{\mathbb{G}})$; $A_m(\rho) \in (0, 1]$ s.t.

$$(7.2) \qquad \begin{aligned} A_m(\rho) &= 1 && \text{if } |\xi| \leq R \\ &= \left(\frac{|\xi|}{R}\right)^m && \text{if } |\xi| \geq R + \eta \end{aligned}$$

with $\eta > 0$ arbitrarily small. Put

$$(7.3) \qquad \tilde{A}_{m,\mu}^{\mathbb{G}} := \frac{1}{k^n} \sum_{\epsilon \in \mathcal{A}^n} \frac{A_m \circ F_\epsilon^n}{|E_\epsilon'|^m};$$

with with $n = n(\hbar, \mu)$ such that

$$(7.4) \qquad e_{\min}^{-n} = \mathcal{O}(1)\hbar^\mu \Leftrightarrow n(\hbar, \mu) = [\mu \frac{\log \hbar^{-1}}{\log e_{min}}].$$

For any point $\rho \notin F^{-n}(Z_{\mathbb{G}})$, we have, by lemma 14 , (5.3) and (7.2), that

$$\tilde{A}_{m,\mu}^{\mathbb{G}}(\rho) := \frac{R^{|m|}}{k^n} \sum_{\epsilon \in \mathcal{A}^n} \left|\xi - S_\epsilon^{\mathbb{G}}(\rho_c)\right|^m.$$

Since, from the proof of lemma 15, we know that $S_\epsilon^{\mathbb{G}}(\rho_c)$ is smooth (uniformly in n), we get directly, with (7.4), that the symbol class estimates (3.2) and (4.9) of $\hbar^{\mu m} S_\mu^m$ are satisfied as long as $\left|\xi - S_\epsilon^{\mathbb{G}}(\rho_c)\right| \geq e_{\min}^{-n} = \mathcal{O}(1)\hbar^\mu$.

By lemma 15 if ρ is at distance at least $(R + 2S_{\max}^{\mathbb{G}})e_{\min}^{-n}$ from $K_{\mathbb{G}}$ than ρ is both out of $F^{-n}(Z_{\mathbb{G}})$ and satisfies $\left|\xi - S_\epsilon^{\mathbb{G}}(\rho_c)\right| \geq e_{\min}^{-n}$. We can always smooth out $\tilde{A}_{m,\mu}^{\mathbb{G}}$ near $K_{\mathbb{G}}$ to define $A_{m,\mu}^{\mathbb{G}} \in \hbar^{\mu m} S_\mu^m$ so that $\tilde{A}_{m,\nu}^{\mathbb{G}} \equiv A_{m,\nu}^{\mathbb{G}}$ out of the neighbourhood $K_{\mathbb{G}}^{\mathsf{C}_0 e_{\min}^{-n}} \supseteq F^{-n}(Z_{\mathbb{G}})$ with $\mathsf{C}_0 = 2(R + 2S_{\max}^{\mathbb{G}})$.

On the other hand, again by lemma 14 and (7.3), out of $F^{-n}(Z_{\mathbb{G}})$ we also have that, for any letter ϵ,

$$\frac{\tilde{A}_{m,\mu}^{\mathbb{G}}(F_{\epsilon_0}(\rho))}{\tilde{A}_{m,\mu}^{\mathbb{G}}(\rho)} \leq \kappa^m \frac{\sum_{\epsilon \in \mathcal{A}^n} |E_\epsilon'(x)|^m}{\sum_{\epsilon \in \mathcal{A}^n} |E_\epsilon'(x_{\epsilon_0})|^m}.$$

Now, for some absolute constant $C > 0$, $e^{nP(|m|)-C} \leq \sum_{\epsilon \in \mathcal{A}^n} |E_\epsilon'(x)|^m \leq e^{nP(|m|)+C}$ with $P(|m|)$ the topological pressure of E associated to the potential $-m \log E'$ (see [6] Theorem 5.1 p. 72). Thus choosing $\mathsf{C}_1 = e^{2C}$ concludes the proof of lemma 17 with $A_{m,\mu}^{\mathbb{G}}$ as the escape function $\qquad \square$

7.2. **End of the proof.**

Since $A_{m,\mu}^{\mathbb{G}}$ is elliptic and of order m the following spaces of distributions

$$\mathcal{H}_{\alpha,\mu}^m := \mathrm{Op}\left(A_{m,\mu}^{\mathbb{G}}\right)^{-1}\left(L^2\left(S^1\right) \otimes \mathcal{D}_\alpha\right)$$

are Hilbert spaces w.r. to the norm inherited from L^2 and are isomorphic in terms of subsets of $\mathcal{D}'\left(S^1\right) \otimes \mathcal{D}_\alpha$ to $H^m\left(S^1\right) \otimes \mathcal{D}_\alpha$. Theorem 5 essentially reduces to the following statement:

Lemma 18. *Choose any* $\mathsf{C} > \mathsf{C}_0$ *and set* $C^{-1}_{U(1)} = 2\pi;\ C^{-1}_{SU(2)} = 8\pi^2.\ \forall \epsilon > 0,$ $\forall 0 \leq \mu < \frac{1}{2},\ m < 0$ *sufficiently negative and* $|\alpha| > 0$ *large enough,*

$$\sharp \left\{ spct \left(\hat{F}_\alpha |_{\mathcal{H}^m_{\alpha,\mu}} \right) \bigcap \mathbb{C} \backslash D^{\mathbb{C}}_\epsilon \right\} \leq C_{\mathbb{G}}\, dim_{\mathbb{C}} \left(\mathcal{D}_\alpha \right) |\alpha|\, Vol \left\{ K^{\mathsf{C}|\alpha|^{-\mu}}_{\mathbb{G}} \right\} (1 + o(1)) .$$

Proof. $\hat{F}_\alpha : \mathcal{H}^m_{\alpha,\mu} \to \mathcal{H}^m_{\alpha,\mu}$ is by construction unitary equivalent to

$$\hat{Q}_{m,\mu} := \mathrm{Op} \left(A^{\mathbb{G}}_{m,\mu} \right) \hat{F}_\alpha \mathrm{Op} \left(A^{\mathbb{G}}_{m,\mu} \right)^{-1} : L^2 \left(S^1 \right) \otimes \mathcal{D}_\alpha \to L^2 \left(S^1 \right) \otimes \mathcal{D}_\alpha.$$

Define

$$\hat{P}_\mu := \hat{Q}^*_{m,\mu} \hat{Q}_{m,\mu} = \mathrm{Op} \left(A^{\mathbb{G}}_{m,\mu} \right)^{-1} \hat{F}^*_\alpha \mathrm{Op} \left(A^{\mathbb{G}}_{m,\mu} \right)^2 \hat{F}_\alpha \mathrm{Op} \left(A^{\mathbb{G}}_{m,\mu} \right)^{-1} .$$

By the composition and Egorov theorems (4.2), (4.10) $\hat{P}_\mu \in OPS^0_\mu$ and its symbol reads

$$P_\mu = \sum_{\epsilon \in \mathcal{A}} \left(\frac{A^{\mathbb{G}}_{m,\mu} \circ F_\epsilon}{A^{\mathbb{G}}_{m,\mu}} \right)^2 \mod \hbar^{1-2\mu} S^{-1}_\mu.$$

From lemma 17, P_μ naturally decomposes into a compact part K_μ supported on $K^{\mathsf{C}\hbar^\mu}_{\mathbb{G}}$, with some $\mathsf{C} > \mathsf{C}_0$, and a bounded part R_μ with $\sup |R_\mu| \leq \mathsf{C}^2_1 \kappa^{2m} + \mathcal{O}(\hbar^{1-2\mu})$. With lemma 7, the decomposition transposes to the operator level with $\hat{P}_\mu = \hat{K}_\mu + \hat{R}_\mu,\ \hat{K}_\mu := \mathrm{Op}_\hbar (K_\mu)$ trace class and self-adjoint and $\left\| \hat{R}_\mu \right\| \leq \mathsf{C}^2_1 \kappa^{2m} + \mathcal{O}(\hbar^{1-2\mu})$. From lemma 20 in the appendix , we have that $\forall \epsilon > 0$ and $\hbar \equiv |\alpha|^{-1}$ small enough

$$\sharp \left\{ \mathrm{spct} \left(\hat{K}_\mu \right) \bigcap \mathbb{R} \backslash (-\epsilon; \epsilon) \right\} \leq f_{\mathbb{G}}(\alpha)|\alpha|\mathrm{Vol} \left\{ K^{\mathsf{C}\hbar^\mu}_{\mathbb{G}} \right\} (1 + o(1)) ,$$

with $f_{U(1)}(\nu) \equiv \frac{1}{2\pi}$ and $f_{SU(2)}(j) = \frac{1}{8\pi^2} dim_{\mathbb{C}} (\mathcal{D}_j)$. By perturbation, eventually choosing a larger $\epsilon > 0$, for m sufficiently negative and $\hbar$ small enough, the same is true for the eigenvalues of $\hat{P}_\mu$ thus for the singular values of $\hat{Q}_{m,\mu}$. Corollary 22 from the appendix allows us to draw the same conclusion for the eigenvalues of $\hat{Q}_{m,\mu}$, yielding the result. $\qquad \square$

Definition 19. The upper Minkowski[5] dimension (or box dimension) of a non empty bounded subset A of $\mathbb{R}^d$ is

$$(7.5) \qquad d - \dim A := \mathrm{co} \dim A := \sup_{s \in \mathbb{R}} \left\{ \limsup_{\delta \downarrow 0} \delta^{-s} \cdot \mathrm{Vol}^d \left(A^\delta \right) < +\infty \right\} .$$

In general $\limsup_{\delta \downarrow 0} \delta^{-co\dim A} \cdot \mathrm{Vol}^d \left(A^\delta \right) < +\infty$ does not hold[6], so $\mathrm{Vol}^d \left(A^\delta \right) = \mathcal{O} \left(\delta^{co\dim A - \eta} \right)$ for any $\eta > 0$. We write the latter $\mathrm{Vol}^d \left(A^\delta \right) = \mathcal{O} \left(\delta^{co\dim A - 0} \right)$.

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[5]For nice sets the Minkowski dimension coincides with the Hausdorff dimension $\dim_H$, but in general $\dim_H A \leq \dim A$.

[6]when it does the set A is said to be of pure dimension see [24] for some comments and further references.

From this definition and lemma 18, Theorem 5 follows rather directly. For any $0 \leq \mu < \frac{1}{2}$, the spectrum of $\hat{F}_\alpha|_{\mathcal{H}^m_{\alpha,\mu}}$ is no other than the Ruelle spectrum of resonances, the latter independent of μ. Thus, for any $\epsilon > 0$, $m < 0$ sufficiently negative, for some $\tilde{C}_\mathbb{G} > 0$ independent of α, m, and for $|\alpha|$ large enough:

$$(7.6) \qquad \sharp\left\{\mathrm{spct}\left(\hat{F}_\alpha|_{H^m \otimes \mathcal{D}_\alpha}\right)\bigcap \mathbb{C}\backslash D_\epsilon^\mathbb{C}\right\} \leq \tilde{C}_\mathbb{G}\dim_\mathbb{C}\left(\mathcal{D}_\alpha\right)|\alpha|^{1-\frac{1}{2}co\dim K_\mathbb{G}+0}$$

In the Abelian case

$$\dim_\mathbb{C}\left(\mathcal{D}_\alpha\right)|\alpha|^{1-\frac{1}{2}co\dim K_\mathbb{G}+0} = |\nu|^{\frac{1}{2}(2-(2-\dim K_\mathbb{G}))+0} = |\nu|^{\frac{1}{2}\dim K_\mathbb{G}+0}\,.$$

In the non-Abelian case,

$$\dim_\mathbb{C}\left(\mathcal{D}_\alpha\right)|\alpha|^{1-\frac{1}{2}co\dim K_\mathbb{G}+0} = \left(\frac{2j+1}{j}\right)j^{\frac{1}{2}\dim K_\mathbb{G}+0}.$$

Thus (7.6) yields Theorem 5.

8. Appendix A. Adapted Weyl type estimates

If $a \in S^0_\mu \cap L^2\left(\mathbb{R}^{2d}\right)$ one has the following important *exact* formula [5]:

$$(8.1) \qquad\qquad \mathrm{tr}\left(\mathrm{Op}^w_\hbar(a)\right) = \frac{1}{(2\pi\hbar)^d}\int a(x,\xi)dxd\xi.$$

For the quantization $\mathrm{Op}_j := \mathrm{Op}^{AW}_j \circ \mathrm{Op}^w_{j^{-1}}$ defined in (4.7), using (4.5) and (8.1) we have for any $a \in S^0_\mu(T^*S^1 \times S^2) \cap L^2\left(T^*S^1 \times S^2\right)$:

$$(8.2) \qquad\qquad \mathrm{tr}\left(\mathrm{Op}_j(a)\right) = \frac{\dim_\mathbb{C}\left(\mathcal{D}_j\right)j}{8\pi^2}\int_{T^*S^1 \times S^2} a(x,\xi,\mathbf{n})dxd\xi d\mathbf{n}$$

Lemma 20. *Let $a \in S^{-\infty}_\mu$ be a real compacly supported symbol. $\forall \hbar > 0$, $\mathrm{Op}^w_\hbar(a)$ is self-adjoint and trace class on L^2. Furthermore, for any $\epsilon > 0$, $\hbar$ small enough:*
$$(8.3)$$
$$\sharp\left\{\mathrm{spct}\left(\mathrm{Op}^w_\hbar(a)\right)\bigcap\mathbb{R}\backslash(-\epsilon;\epsilon)\right\} = \frac{1}{(2\pi\hbar)^d}\left(\mathrm{Vol}\left\{|a| > \epsilon\right\} + \mathrm{Vol}\left\{|a| > 0\right\}o(1)\right)$$

The same holds true for $a \in S^{-\infty}_\mu \cap C^\infty_0\left(T^*S^1 \times S^2\right)$, for $j > 0$ large enough, with
$$(8.4)$$
$$\sharp\left\{\mathrm{spct}\left(\mathrm{Op}_j(a)\right)\bigcap\mathbb{R}\backslash(-\epsilon;\epsilon)\right\} = \frac{\dim_\mathbb{C}\left(\mathcal{D}_j\right)j}{8\pi^2}\left(\mathrm{Vol}\left\{|a| > \epsilon\right\} + \mathrm{Vol}\left\{|a| > 0\right\}o(1)\right)$$

Proof. Consider 1_ϵ −the characteristic function of $\mathbb{R}\backslash(-\epsilon;\epsilon)$− so that

$$\mathrm{tr}1_\epsilon\left(\hat{A}\right) = \sharp\left\{\mathrm{spct}\left(\hat{A}\right)\bigcap\mathbb{R}\backslash(-\epsilon;\epsilon)\right\}.$$

Let $P_\pm$ be polynomials of degree N vanishing at zero and approximating 1_ϵ on $[-C;C]$, with $C > \sup|a|$, s.t. $P_-(t) \leq 1_\epsilon(t) \leq P_+(t)$ for any $t \in [-C;C]$. Let us write $\hat{A}$ for $\mathrm{Op}^w_\hbar(a)$. $P_\pm\left(\hat{A}\right)$ are well defined PDOs in $OPS^{-\infty}_\mu$ and, by the

composition theorem, their respective symbols read $P_\pm(a) + \hbar^{2(1-2\mu)}b_\pm$ with $b_\pm$ negligible out of the support of a. By eq.(8.1) :

$$\mathrm{tr} P_\pm\left(\hat{A}\right) = \frac{1}{(2\pi\hbar)^d}\int\left(P_\pm(a) + \hbar^{2(1-2\mu)}b_\pm\right)dxd\xi.$$

Now $\left|\int b_\pm dxd\xi\right| \le \mathrm{Vol}\{|a| > 0\}\, C_N$ with C_N independent of $\hbar$. On the other hand $P_\pm = 1_\epsilon + r_\pm$ on $[-C; C]$ with $r_\pm(0) = 0$. We thus get $\int P_+(a)dxd\xi \le \mathrm{Vol}\{|a| > \epsilon\} + \mathrm{Vol}\{|a| > 0\}\,\delta_N$ and the opposite inequality for $\int P_-(a)dxd\xi$, with $\delta_N \to 0$ as N grows. By the spectral and L^2-continuity (lemma 7) theorems, for $\hbar > 0$ small enough,

$$\mathrm{tr} P_-\left(\hat{A}\right) \le \mathrm{tr} 1_\epsilon\left(\hat{A}\right) \le \mathrm{tr} P_+\left(\hat{A}\right)$$

so

$$\frac{1}{(2\pi\hbar)^d}(\,\mathrm{Vol}\{|a| > \epsilon\} - \mathrm{Vol}\{|a| > 0\}\,(\delta_N + \hbar^{2(1-2\mu)}C_N)\,) \le \mathrm{tr} 1_\epsilon\left(\hat{A}\right)$$

and

$$\mathrm{tr} 1_\epsilon\left(\hat{A}\right) \le \frac{1}{(2\pi\hbar)^d}(\,\mathrm{Vol}\{|a| > \epsilon\} + \mathrm{Vol}\{|a| > 0\}\,(\delta_N + \hbar^{2(1-2\mu)}C_N)\,).$$

As long as $\mu < \frac{1}{2}$, for any $\delta > 0$ arbitrarily small, one can take N large enough s.t. $\delta_N \le \delta/2$ and then $\hbar$ small enough s.t. $\hbar^{2(1-2\mu)}C_N \le \delta/2$, so that the term $\delta_N + \hbar^{2(1-2\mu)}C_N$ is smaller than δ. This gives (8.3). The exact same argument can be carried out for $\hat{A} := \mathrm{Op}_j(a)$, using (8.2) to compute the trace of $P_\pm\left(\hat{A}\right)$ and with $\hbar \equiv j^{-1}$ to get (8.4). $\qquad\square$

9. Appendix B. General lemmas on singular values

Let $(P_\nu)_{\nu\in\mathbb{N}}$ be a family of compact operators on some Hilbert space. Consider any P_ν and let $(\lambda_{j,\nu})_{j\in\mathbb{N}^*} \in \mathbb{C}$ be the sequence of its eigenvalues ordered decreasingly according to multiplicity:

$$|\lambda_{1,\nu}| \ge |\lambda_{2,\nu}| \ge \dots$$

In the same manner, define $(\mu_{j,\nu})_{j\in\mathbb{N}^*} \in \mathbb{R}^+$, the decreasing sequence of singular values of P_ν (the eigenvalues of $\sqrt{P_\nu^*P_\nu}$). Finally let $[x] \in \mathbb{N}$ stand for the integral part of $x \in \mathbb{R}$.

Lemma 21. *Suppose there exits a map $N : \mathbb{N} \to \mathbb{N}$ s.t. $N(\nu) \to \infty$ and $\mu_{N(\nu),\nu} \to 0$ as ν grows. Then $\forall C > 1$, $|\lambda_{[C.N(\nu)],\nu}| \to_{\nu\to\infty} 0$.*

Corollary 22. *Let $N : \mathbb{N} \to \mathbb{N}$ be as in lemma 21. Suppose that $\forall \epsilon > 0$, $\exists A_\epsilon \ge 0$ s.t. $\forall \nu \ge A_\epsilon$; $\#\{j \in \mathbb{N}^* \mid \mu_{j,\nu} > \epsilon\} < N(\nu)$. Then for any $C > 1$, $\epsilon > 0$ there exists $B_{C,\epsilon} \ge 0$ such that:*

$$(9.1) \qquad \forall \nu \ge B_{C,\epsilon};\ \#\{j \in \mathbb{N}^* \mid |\lambda_{j,\nu}| > \epsilon\} \le [C \times N(\nu)].$$

Proof. (Of corollary 22). Suppose that for any $\epsilon > 0$, there exists a rank A_ϵ s.t. for all $\nu \geq A_\epsilon$ $\#\{j \in \mathbb{N}^*$; $\mu_{j,\nu}\} < N(\nu)$, which means that $\mu_{N(\nu),\nu} \to_{\nu \to \infty} 0$ and from Lemma 21, $\forall C > 1$, $|\lambda_{[CN(\nu)],\nu}| \to_{\nu \to \infty} 0$, which can be directly restated as (9.1). $\qquad\square$

Let us now prove the lemma.

Proof. (Of lemma 21) The main relation between singular and eigenvalues is given by the Weyl inequalities (see [11] p. 50 for a proof):

$$(9.2) \qquad \prod_{j=1}^{k} \mu_{j,\nu} \leq \prod_{j=1}^{k} |\lambda_{j,\nu}|; \quad \forall k \in \mathbb{N}^*.$$

Let $m_{j,\nu} := -log\,(\mu_{j,\nu})$, $l_{j,\nu} := -log\,(|\lambda_{j,\nu}|)$ to define $S_{k,\nu} := \sum_{j=1}^{k} m_{j,\nu}$, and $L_{k,\nu} := \sum_{j=1}^{k} l_{j,\nu}$. The Weyl inequalities (9.2) thus reads: $S_{k,\nu} \leq L_{k,\nu}$, $\forall k \in \mathbb{N}^*$. Notice that both sequences $(l_{j,\nu})_{j \geq 1}$ and $(m_{j,\nu})_{j \geq 1}$ are increasing so, $\forall k \in \mathbb{N}^*$, $k \cdot l_{k,\nu} \geq L_{k,\nu}$, and for any $k, K \in \mathbb{N}^*$,

$$(9.3) \qquad S_{k+K,\nu} \geq K \cdot m_{k,\nu}.$$

Suppose that $\mu_{N(\nu),\nu} \to 0$ (hence $m_{N(\nu),\nu} \to \infty$) as $\nu \to \infty$ and choose some constant $C > 1$, By (9.3) we have that

$$(9.4) \qquad S_{[CN(\nu)],\nu} \geq ([CN(\nu)] - N(\nu)) \cdot m_{N(\nu),\nu},$$

and therefore, since $l_{[CN(\nu)],\nu} \geq \frac{1}{[CN(\nu)]} \times L_{[CN(\nu)],\nu} \geq \frac{1}{[CN(\nu)]} \times S_{[CN(\nu)],\nu}$, from (9.4) we get

$$(9.5) \qquad l_{[CN(\nu)],\nu} \geq \frac{[CN(\nu)] - N(\nu)}{[CN(\nu)]} \times m_{[CN(\nu)],\nu}.$$

Notice that $[CN(\nu)] - N(\nu) > 0$ for ν large enough. Therefore (9.5) gives the result. $\qquad\square$

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